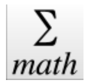
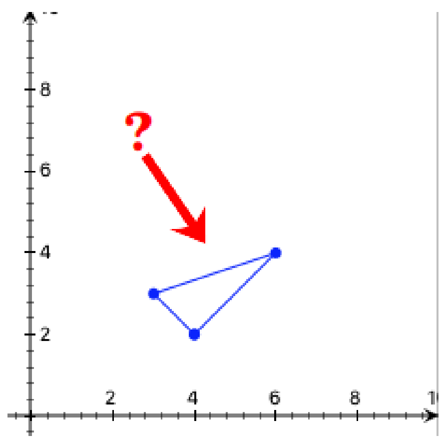
Exploring Math  with EIGENMATH

Linear Algebra *Interactive*! with Eigenmath

Part 5

Determinants



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About this Booklet

This is part 5 of a series of booklets, which want to introduce the reader to some topics of elementary Linear Algebra and at the same time into the use of CAS EIGENMATH.
This booklet grew out from a series of papers that I developed around 2000 mainly at the University of Duisburg in Germany, but it is revised, renewed and adapted to EIGENMATH. It is based on an educational study, which I carried out as part of a high school experiment for mathematics lessons with the use of computer algebra systems (CAS) in the state of North Rhine-Westphalia (NRW) and was published in [6]. The material was repeatedly tested at a German hight school resp. college. The learning parcourses were originally developed using notebooks compiled with various versions of the CAS DERIVE and accompanying learning materials in paper form.

About the content of the booklet

The compact solution formula of G. CRAMER for regular linear systems of equations is explored and gradually programmed in EIGENMATH. The analysis of the associated solution process naturally leads to the development of the cofactor concept of a determinant and to the adjugate of a matrix. We then gain a dimension-independent formula for the CRAMER rule and a deeper insight into the structure of the inverse of a matrix. Specializations and EIGENMATH experiments in the associated collection of exercises discuss, among other things, the wedge product of two–dimensional vectors (with application to the intersection formula for straight lines in the plane) as well as the cross and the Box product of three–dimensional vectors.
The often isolated introduction of the mentioned concepts (*cross* product, spar alias Box alias triple product) is avoided and here genetically arises from the investigation of the linear system solution process. The teaching units offers a geometrically oriented alternative to the treatment of linear system via the GAUSS-JORDAN method. At the same time, algebraic and geometric insights are linked, since the special 3D CRAMER rule turns out to be geometrically interpretable as the ratio of the volumina of two parallelepipeds.
An interdisciplinary aspect occurs through the use of elementary methods of software engineering in the bottom-up development and step-by-step refinement of the diverse `Cramer` functions. Techniques of this kind can often be used in CAS and train algorithmic oriented constructive thinking. The EIGENMATH commands used and the textual representation should be elementary enough to serve as a good companion while reading basic or advanced courses on Linear Algebra or as a help system for independent individual work.

A short sketch of the APOS learning theory

The social-constructivist APOS[1] learning theory was in my mind throughout the construction of these booklets: as a theoretical research approach, for the practical curriculum development and as a computer-aided, cooperative teaching–learning method. Compared to classic learning theories, the APOS theory focuses on the finding that *the mental*

[1] see for example ARNON et. al. [1] or my thesis for a German introduction [9, pp.16–48]

(re)construction process of mathematical knowledge is decisively promoted by a mathematically –oriented programming language as a medium in which the knowledge constructions are represented as programming constructs (Dubinsky). Starting with the epistemological reflection of a mathematical concept with the aim of a 'genetic decomposition' of the concept in question, specific mental constructions are suggested that a learner needs to acquire the concept and these are represented in the CAS. The learning process is triggered by *actions* or manipulations on mental or virtual CAS objects (actions); these actions are interiorized ('internalized') by the learning subject into *processes*, that are finally encapsulated in the form of *objects*. It should be possible to decompress such objects back into the processes from which they were constructed. Processes or objects are thematically networked and structured in the form of *schemas*, stored in the learner's knowledge network - hence the acronym A.P.O.S.
In the A.P. phase the individual learning trajectories of the learners meander around the hypothetical learning trajectory, which was designed by the instructor resp. teacher. An "object" understanding of a mathematical concept may also be interpreted as a *concept definition* and an "schema" forming as a *concept image* in terms of Tall and Vinner, [17]. In the APOS theory, *learning difficulties* are preferably explained with an unsuccessful interiorization of actions into processes or the failed encapsulation of processes into objects or an inadequate structuring of objects into a schema.
The chapters of the booklet also partially represent so-called '*microworlds*' (e.g. model problems) in which a local mathematical knowledge domain with its manipulable objects (here: matrices) and operations (here: dot(), adj(), etc.) is mapped into the language of the CAS Eigenmath.

Eigenmath

The considerations in this script would be difficult to elementarize without the use of a computer algebra system like Eigenmath, because heavy calculations of products and inverses of matrices occur in the conceptual constructions. Therefore, in Eigenmath laboratories we explore decisive phenomena or verify or falsify hypotheses and would like to encourage ongoing dialogical practice in CAS language communication skills with the Eigenmath assistance.
The accompanying colloquial comments are deliberately short. If possible, all CAS dialog sequences - which are shown in `typewriter font` - should be performed live on the computer. We give therefore many lively links to invocable Eigenmath scripts. The Eigenmath routines, which are written for this Part 5, are collected in the toolbox `craBox.txt` for the convenience of the user and are invoked by the command `run("craBox.txt")` in a running Eigenmath Online[2] session. In this way you can s(t)imulate this communication process at the Eigenmath prompt region in the input ("Run") window and allow a dynamic interactive 'reading act' with spontaneous deviations, additional inquiries or ad hoc explorations, which would otherwise be not possible.

[2]Running the Eigenmath app on the iMac this command has to be substituted trough `run("downloads/craBox.txt")`. The file `craBox.txt` has therefore to be copied to the 'downloads' folder.

EIGENMATH is a computer algebra system that can be used to solve problems in mathematics and the natural and engineering sciences. It is a personal resource for students, teachers and scientists. EIGENMATH is small, compact, capable and free. It runs on WindowsOS, MacOS, Android and online in a browser. It is in the opinion and experience of the author very well suited for doing linear algebra from the viewpoint of APOS theory.

To use this booklet interactively
... *you do not need to install any software to do the calculations*! The CAS EIGENMATH works directly out of this text, on any operating system, on every hardware (Smartphone, iPhone, tablet, PC, etc.), at any place: you only must be online and click on a link like ▷*Click here to invoke* EIGENMATH (◁ please click here! Really!). From this point on you can run a given script or fork with own computations.
... *you do not need to install any software to produce quality plots interactively*! You only must be online to press a link like `CalcPlot3D` (◁ please click here! Really!) in this script. At this point you can make a 2D/3D–plot to visualize a concept or to make a calculation visually evident.

I thank George WEIGT for his friendly support, hints and help regarding his EIGENMATH. So it was a real pleasure to write down these notes.

Any feedback from the user is very welcome.

PS: Being retired and no native speaker, I have no support from colleges at high school or university anymore, therefore the reader may excuse me for my grammatical and spelling mistakes.

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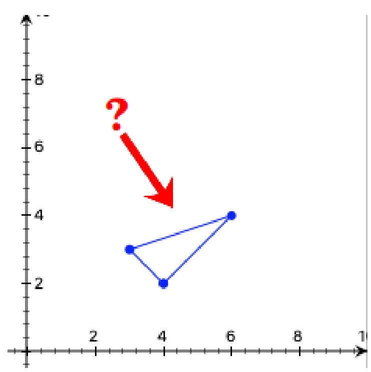
February 2021

15 Cramer rule, Determinants and Products

If a system of linear equations $A * X = B$ is *regular*[3], then we know that one can find the unique solution X through the well-known formula $X = A^{-1} * B$, i.e. the system of linear equations is multiplied for *the unique solution* on both sides with the inverse matrix A^{-1}. In part 3 of this series we looked at the solution process from a constructive computational point of view using the Gauss-Jordan algorithm and the associated Eigenmath procedure `RREF` to solve such equations and to describe the solution set.
In this chapter we look at the solution process from an algebraic point of view using determinants as main means. By the way we network diverse algebraic concepts like the *adjoint, cofactor*s and *minor*s and we obtain a new handy solution formula for regular linear equations: the famous so–called Cramer rule, which allows a nice geometrical interpretation.

Let's start an first exploration in 3 steps.

15.1 Preimage I – the 2D Cramer rule



How to calculate the preimage of a figure under an reversible linear mapping?
Or: From which unknown original point X does point $P = (6, 4)$ come from under the map $M : (x, y) \mapsto (3x + 2y, x + 2y)$?
Short in matrix language:

$$\begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} * \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix} \qquad \text{with} \quad x =?, y =?$$

Shorter:

$$M * X = P \qquad \text{with} \quad X = [x, y] =?$$

15.1.1 calculate concrete example preimage

We calculate the unknown point X with $X \overset{M}{\mapsto} P$ using the inverse matrix method.

[3]e.g. unique solvable, that means the determinant of A is non-zero.

That is: Since the linear mapping is reversible because of its non-vanishing determinant, we solve by multiplying the matrix equation from the left with the inverse mapping matrix. Here is the EIGENMATH calc sheet:

```
M=((3,2),(1,2))
P=(6,4)

X=dot(inv(M),P)
X                              -- returns (1, 1.5)

dot(M,X)                       -- check ok
```

Try it: ▷*Click here to run the calc sheet.*

Exercise 15.1. Read off the coordinates of the two other points Q, R of the triangle $\triangle PQR$ and determine their unknown preimage points Y, Z, too. Use it to complete the sketch around the starting triangle in CALCPLOT3D

Exercise 15.2. Set up an EIGENMATH solution, which calculates the complete preimage $\triangle XYZ$ simultaneously in one equation. Plot both triangles wit CALCPLOT3D.
▷*Look up solution sheet.* EIGENMATH output:

$$X = \begin{bmatrix} 1 & 0 & 1 \\ \frac{3}{2} & \frac{3}{2} & \frac{1}{2} \end{bmatrix}$$

15.1.2 analyze of the solution process with a general matrix

We now study a generalization of the situation in order to gain a general pattern. In order to be able to follow the solution process in this laboratory in detail, we have to replace the specific numerical values with placeholders, so that interim calculations become visible and the results can be analyzed.

$$M * X = P$$
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} * \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix} \qquad \text{with} \quad x = ?, y = ?$$

We assume, that $\det(M) \neq 0$ and determine a 'general' solution for X using EIGENMATH's build–in function `inv(.)` to calculate the inverse matrix of M. Think about the output!

```
M = ((a,b),(c,d)) -- general matrix

P = (p,q)         -- image point

inv(M)            -- M^(-1)

X=dot(inv(M),P)   -- M^(-1)*P
X                 -- preimage point
```

$$\begin{bmatrix} \frac{d}{ad - bc} & -\frac{b}{ad - bc} \\ -\frac{c}{ad - bc} & \frac{a}{ad - bc} \end{bmatrix}$$

$$X = \begin{bmatrix} -\frac{bq}{ad - bc} + \frac{dp}{ad - bc} \\ \frac{aq}{ad - bc} - \frac{cp}{ad - bc} \end{bmatrix}$$

We see some patterns evolve. First all denominators as well in M^{-1} as in the solution point X has the same term $ad - bc$. How is the term of the denominator to be interpreted? We invoke EIGENMATH's `det` function to look at his general term:

```
M = ((a,b),(c,d))

det(M)                                  --(1)

run("downloads/gjBox.txt")
Ge(M)
DET = a*(-b*c/a+d)                      --(2)
DET
```

$$a\,d - b\,c$$

$$\begin{bmatrix} a & b \\ 0 & -\frac{b\,c}{a} + d \end{bmatrix}$$

$$D_{ET} = a\,d - b\,c$$

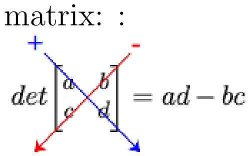
We see in (1), that EIGENMATH returns the term $ad - bc = det(M)$. To verify, that this term is in coincidence with our definition of `Det` we row-reduce matrix M via `Ge`, i.e. via the GAUSS algorithm. According to our definition we then have to multiply the diagonal elements in (2), which results in the same term. Ok.
We draw two consequences.

1. The term for the inverse of M $\underset{\text{EIGENM: } \texttt{inv(M)}}{\textit{Math}: M^{-1}}$ can be written as $\frac{1}{det(M)} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

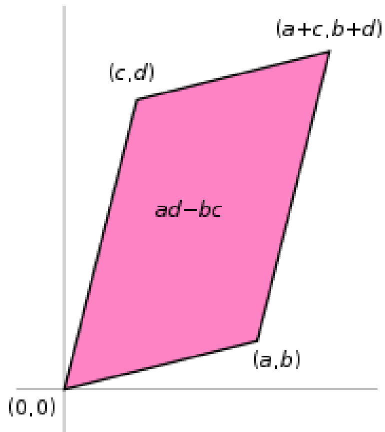
2. The x–coordinate of the solution vector $X = (x, y)$ is

$$\begin{aligned} x &= \frac{-bq + dp}{ad - bc} = \frac{dp - bq}{det(M)} \\ &= \frac{\det \begin{bmatrix} p & b \\ q & d \end{bmatrix}}{\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}} \end{aligned} \qquad \text{(Cr22)}$$

Try it: ▷*Click here to run the script.*
○ Consider the result (Cr22): how were the numerator and denominator systematically formed from the 6 data a, b, c, d, p, q? The result is the so-called CRAMER rule:
• *The 1^{st} coordinate x of the solution point $X = [x, y]$ can be calculated as the quotient of two determinants*: the denominator is the determinant of the matrix A and the numerator is the determinant of the modified matrix, in which the 1st column of the matrix was replaced by the given point (= right side of the equation).
• The 2^{nd} coordinate y of the solution point $X = [x, y]$ is also obtained as the ratio of two determinants: the denominator is again the determinant of the matrix A and the numerator is the determinant of the modified matrix, in which this time the 2nd column has been replaced by the given point.
• By the way, we have the following mnemonic pattern, the so-called *rule of* LEIBNIZ, to calculate the determinant of a 2×2 matrix: :



Exercise 15.3 (Geometrical interpretation of determinant as area of a parallelogram).



The LEIBNIZ formula for the determinant $det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$ of a 2×2 matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ allows a geometrical interpretation. Show:
Let the columns $A = \overrightarrow{OA} = (a, b)$ and $C = \overrightarrow{OC} = (c, d)$ of matrix M be drawn as the sides of a parallelogram[4] $OABC$ with vertices at $O = (0, 0), A = (a, b), B = (a + c, b + d)$, and $C = (c, d)$. Verify by an elementary geometric argument, that *the (oriented) area of parallelogram* $OABC$ *equals* $det(A)$, i.e.

$$area(OABC) = |det(\overrightarrow{OA}, \overrightarrow{OC})|$$

• Therefore we can interpret formula (Cr22) now in geometric language: *The* 1^{st} *coordinate* x *of the solution point* $X = [x, y]$ *of the regular 2×2 linear system* $A * X = B$ *is the ratio of two parallelogram areas.*

Exercise 15.4. Calculate the solution of the linear system in Ex.15.1 using 2-dimensional determinants, i.e. the rule of CRAMER and the LEIBNIZ formula.

> *The calculation of the solutions of a linear system of equations as the quotient of two determinants is a new, unexpected solution method that we will study in more detail below and also automate with the help of* EIGENMATH. *We also strive for a geometric understanding. That is the aim of the following section.*

15.1.3 automatize the solution process – the 2×2 CRAMER rule

In order to calculate the solution of regular 2×2 linear systems of equations fully automatically with EIGENMATH, we must be able to carry out the observed column replacement process in formula (Cr22), when modifying the system matrix M. To do this, we create a new EIGENMATH command `Replace`.

```
# EIGENMATH is row oriented.
```

[4] We cite the figure from `https://en.wikipedia.org/wiki/Determinant`

```
# But we have to Replace the 1st column.
# Therefore we must transpose and re-transpose.

Replace(pp,M,i) = do(X = M,                   --(1)
                     Xt = transpose(X),       --(2)
                     Xt[i] = pp,              --(3)
                     transpose(Xt))           --(4)

# implement the 2x2 Cramer rule (Cr22)              --(5)
Cramer22(A,B) = ( det(Replace(B,A,1))/det(A),
                  det(Replace(B,A,2))/det(A) )

---             (6)             (7)
c22 = Cramer22( ((3,2),(1,2)),  (6,4) )
c22
```

Comment. To replace the i–th column of M with the RHS pp of the linear system, we first (1) save a copy of matrix M in the container variable X. Then (2) we transpose X alias M to focus on the column structure of M. In step (3) we transfer the RHS pp into the i–th column $Xt[i]$ of X. Step (4) returns the transposed transpose, i.e. the original row structure. The implementation of the CRAMER rule in (5) follows 1:1 the mathematical observation in formula (Cr22), where the first line calculates the x component of the solution as a fraction with nominator equals the determinant of the matrix `Replace(B,A,1)`, i.e. the matrix A, whose 1st column is replaced with the RHS B.
In $c22$ we see an example invocation of function `Cramer22` with the mapping matrix and the image point of 15.1. The result is $c22 = (1, 1.5) = X$.
Try it: ▷*Click here to run the example above.*

⋈

P133. CRAMER **rule I.** For an first exercise, solve the following 2×2 linear systems I, II and III using the CRAMER rule by paper'n pencil and controll your calculation by EIGENMATH:

$$\mathrm{I} := \begin{bmatrix} 1\cdot x + 2\cdot y = 3 \\ 4\cdot x + 5\cdot y = 6 \end{bmatrix} \qquad \mathrm{II} := \begin{bmatrix} -6\cdot x - 5\cdot y = -4 \\ -3\cdot x - 2\cdot y = -1 \end{bmatrix}$$

$$\mathrm{III} := \begin{bmatrix} \frac{1}{1}\cdot x + \frac{1}{2}\cdot y = \frac{1}{3} \\ \frac{1}{4}\cdot x + \frac{1}{5}\cdot y = \frac{1}{6} \end{bmatrix}$$

P134. CRAMER **rule II.**
Consider again the system of linear equations

$$\begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} * \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

a. Interpret the determinant of the mapping matrix M as area of the parallelogram, whose sides are spanned by the columns of M. Do the same for the nominator in CRAMER rule formula. Draw both parallelograms in CALCPLOT3D's graphics window and try to "read" the solution $x = 1$ off the graphic.
To what extent does the x–coordinate of the solution point appear as an area ratio, when changing the mapping matrix with the result point $[6, 4]$?
b. Similarly to a), determine the y–coordinate of the solution point graphically.

P135. Solution calculations.
a. Under what conditions is the CRAMER rule formula applicable?
In the case of inapplicability, add an output of the form "LS with CRAMER rule not solvable" to the EIGENMATH function formula.
b. Choose some problems from your textbook to solve 2×2 – LS with CRAMER22.
Occasionally check the result with a paper'n pencil calculation.

P136. Solution invariance.
The LS $\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} * \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$ has the unique solution $x = -3$ and $y = 3$.
Does the solution change, when the LS is multiplied with the matrix $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ from the left?
What is the geometric effect of this matrix?

P137. Parameter dependency.
For which values of k does the following 2×2 – LS have a unique solution?

$$\begin{bmatrix} 1 & 2 \\ 1 & k \end{bmatrix} * \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

P138. Line fitting.
If one wants to lay a straight line $y = a_0 + a_1 \cdot x$ through two points $P_1 = [x_1, y_1]$ and $P_2 = [x_2, y_2]$, one has to find a solution $[a_0, a_1]$ of the two linear equations

$$\begin{bmatrix} a_0 + a_1 x_1 = y_1 \\ a_0 + a_1 x_2 = y_2 \end{bmatrix}$$

a. Explain, that a solution exists if and only if $\det(\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix}) \neq 0.$

b. Which line goes through [2,1] and [5, -1]?

c. Draw a figure with CALCPLOT3D to verify your solution.

15.1.4 The *wedge* product of two vectors in the plane $\mathbb{R}^2$

For two vectors $A = [a_1, a_2]$ and $B = [b_1, b_2]$ of the plane, one defines its so-called *wedge product* $A \wedge B$ (read: 'A wedge B') as the real number defined by the formula:

$$A \wedge B \overset{def}{=} \det[A, B] = a_1 \cdot b_2 - a_2 \cdot b_1$$

Note: $\det[A, B]$ is meant as the determinant of the matrix $M = [A, B]$, whose columns (or rows) are the vectors A and B.

Exercise 15.5. Implement an EIGENMATH function `wedge(A,B)`, which returns the value $A \wedge B$.

a. Calculate with and without EIGENMATH:

- $[1, 2] \wedge [2, 4] =?$
- $[1, 2] \wedge [1, 2]$
- $2 \cdot [-2, 4] \wedge [1, 0]$.

b. Show (maybe using EIGENMATH), that the name wedge *product* is justified, because – among other things – the following rules apply.

The wedge product is

- *homogeneous*: $A \wedge (kB) = k(A \wedge B)$ with $k \in \mathbb{R}$
- *distributive*: $A \wedge (B + C) = (A \wedge B) + (A \wedge C)$

In contrast to the 'normal' multiplication of numbers, however, the following applies:

- *anticommutative*: $A \wedge B = -(B \wedge A)$
- *alternating*: $A \wedge A = 0$
- Does: $A \wedge (B \wedge C) = (A \wedge B) \wedge C$?

Give a numerical example for each of the properties of the wedge product.

c. Find and prove more laws.

d. Show: the area F of the triangle $\triangle ABC$ with $A = [a_1, a_2], B = [b_1, b_2], C = [c_1, c_2]$ is

$$2 \cdot F = (A \wedge B) + (B \wedge C) + (C \wedge A)$$

Calculate the area of triangle $\triangle[(1|1), (4, 2), (3|5)]$ according to d.
Verify the result with a calculation by paper'n pencil and a quality plot with CALCPLOT3D.

e. Find $Q = (x, y)$, such that the triangle $\triangle OPQ$ with $O = (0, 0), P = (4, 1)$ has the area $F = 5$. Give all solution points Q.

f. Write the 2×2 CRAMER rule using the wedge product notation, i.e. show for $A * X = B$:

$$x = \frac{B \wedge A_1}{A_1 \wedge A_2}$$

where the matrix $A = [A_1, A_2]$ has rows A_1 and A_2. Derive an formula for y.

g. Implement an EIGENMATH function `Cramer22wedge(A,B)`, which returns the complete solution vector (x, y) of an regular 2×2 –linear system $A * X = B$ using the wedge product.

Remark.
The wedge product is sometimes also called the *outer product* or *cap*.
Note: *the wedge product is a real number, that equals to the determinant value; however, their factors are interpreted as isolated vectors and not as a 2×2 matrix!*

> *The wedge product has numerous applications in elementary geometry and in computer graphics for generating direct solution formulas. For example:*

P139. Intersection formula: line–line–intersection.
Two straight lines g and h are given in the plane $\mathbb{R}^2$ by two points on each of them, i.e. $A, B \in g$ and $C, D \in h$, then their point of intersection S_{gh} is calculated using the following explicit formula

$$Sgh(A, B, C, D) = \frac{(C \wedge D) \bullet (B - A) - (A \wedge B) \bullet (D - C}{(B - A) \wedge (D - C)}$$

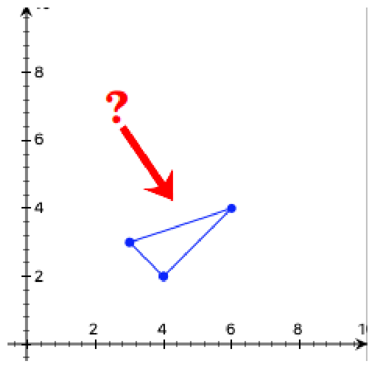
a. Implement the intersection formula in EIGENMATH as function `Sgh(A,B,C,D)`.

b. Test the intersection formula on self-chosen examples.
Verify your results using a figure with CALCPLOT3D.

c. Under what condition does no intersection exist?
Interpret the condition geometrically!

∘ Argue: vectors, whose wedge product is zero, are linearly dependent.

15.2 Preimage II – the 3D Cramer rule



How to calculate the preimage of a figure under an reversible affine mapping?
Or: From which unknown original point X does point $P = (6,4)$ come from under the map $M : (x,y) \mapsto (3x + 2y - 1, x + 2y + 2)$?
We formulate the map M in matrix language, with the added 'translation' vector $T = \left[\begin{smallmatrix}-1\\2\end{smallmatrix}\right]$:

$$M * X \qquad + T \qquad = P \qquad \text{with} \quad X = [x,y]$$

$$\begin{bmatrix}3 & 2\\1 & 2\end{bmatrix} * \begin{bmatrix}x\\y\end{bmatrix} + \begin{bmatrix}-1\\2\end{bmatrix} = \begin{bmatrix}6\\4\end{bmatrix} \qquad \text{with} \quad x = ?, y = ?$$

Verify, that this equation can be written as a 3D matrix equation $A * X = B$, where the translation vector of the map M is integrated in a 3×3 matrix:

$$A * X = B \qquad \text{with} \quad X = [x,y,1]$$

$$\begin{bmatrix}3 & 2 & -1\\2 & 1 & 2\\0 & 0 & 1\end{bmatrix} * \begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}6\\4\\1\end{bmatrix} \qquad \text{with} \quad x = ?, y = ?, z = 1$$

This 'trick' is called a *homogenization*[5] of the affine map or a *lifting* into $\mathbb{R}^3$. We now try to solve this 3×3 linear system with an adapted Cramer rule.

15.2.1 Solution of 3×3 linear systems by modified Cramer rule

In order to calculate the solution of this regular (argue!) 3×3 linear systems of equations, we make a conclusion by analogy and lift our Cramer rule also in the 3rd dimension using the recipe of the column replacement process. The Eigenmath command `Replace` remains unchanged and we try:

```
# CRAMER 3x3 rule
Replace(pp,M,i) = do(X = M,
                     Xt = transpose(X),
```

[5] i.e. we transform the affine map M into a linear one. See my booklet about Linear Transformations with Eigenmath.

```
                    Xt[i] = pp,
                    transpose(Xt))

Cramer33(A,B) = ( det(Replace(B,A,1))/det(A),
                  det(Replace(B,A,2))/det(A),
                  det(Replace(B,A,3))/det(A) )

A = ((3,2,-1),(2,1,2),(0,0,1))
B = (6,4,1)
c33 = Cramer33(A,B)
c33
```

EIGENMATH output: $c33 = (-3, 8, 1) = X$.
Try it: ▷*Click here to run the example above.*

Exercise 15.6. **a.** Check the result using the direct method $X = A^{-1} * B$.
b. Solve the following 3×3 linear system with EIGENMATH function `Cramer33(..)` ... and by paper'n pencil.

$$\begin{bmatrix} 2 & 2 & 4 \\ 1 & 2 & -5 \\ 3 & 1 & -3 \end{bmatrix} * \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 5 \end{bmatrix}$$

15.2.2 Diving deeper into the 3×3 CRAMER rule

Solving Ex.15.6.b by paper'n pencil turned out to be very troublesome, because of the calculation of the many 3×3 determinants! So we ask:

is there an analogy to the 2×2 LEIBNIZ rule $det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$?

Let's start an exploration with EIGENMATH.[6]

`A = ((3,2,-1),(2,1,2),(0,0,1))` `det(A)`	
`A=((2,2,4),(1,2,-5),(3,1,-3))` `det(A)`	-1
`do(i=quote(i), e=quote(e))      --(1)` `A=((a,b,c),(d,e,f),(g,h,i))` `det(A)`	-46 $a\,e\,i - a\,f\,h - b\,d\,i + b\,f\,g + c\,d\,h - c\,e\,g$

▷*Click here to run the example.*

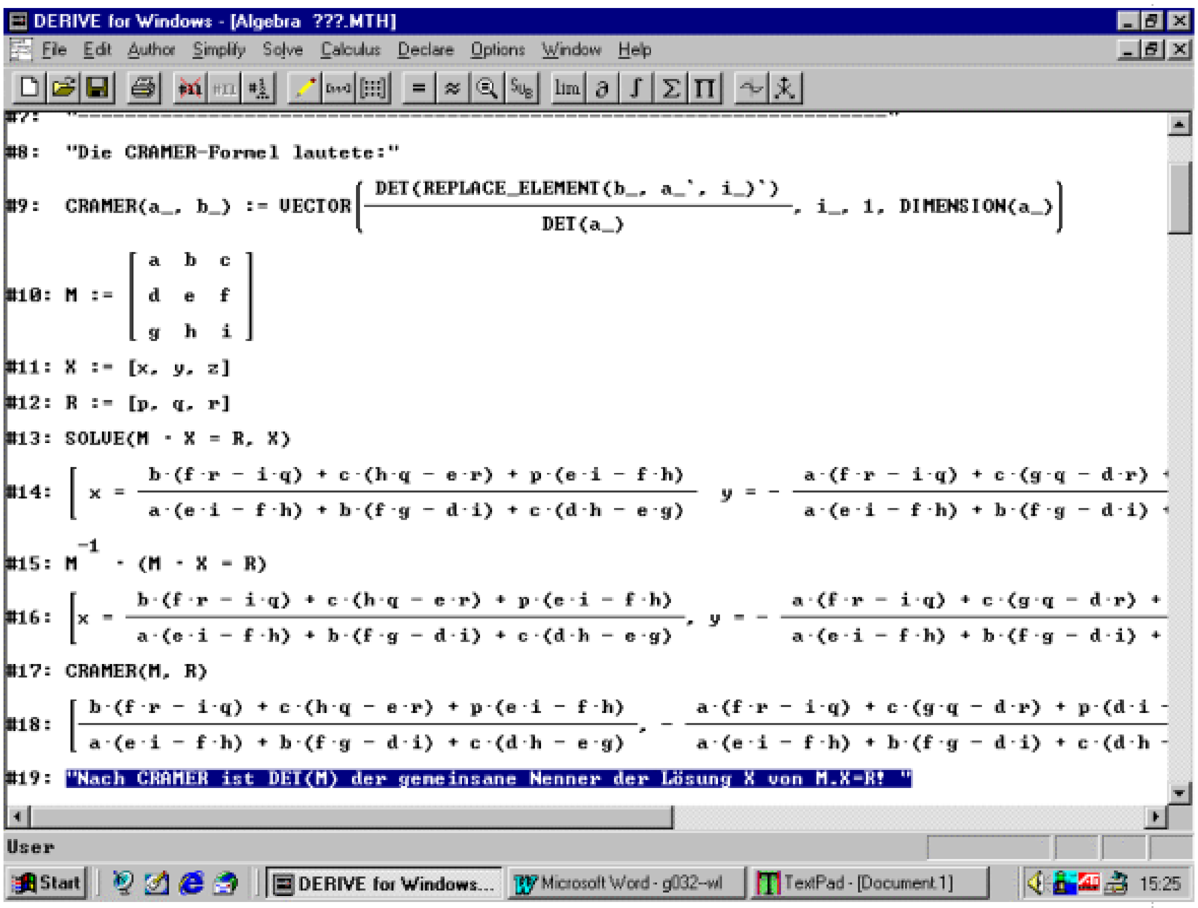
The first two calculations verify, that the linear systems from 15.2.1 are indeed regular, because of $det(A) \neq 0$. To get the determinant of a general 3×3 matrix, EIGENMATH returns a 6 summands formula, the 3×3 LEIBNIZ rule:

$$det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = aei - afh - bdi + bfg + cdh - ceg \qquad (Leibniz3)$$

[6] In the EIGENMATH script we run the command (1) to quote the identifiers $e = exp(1)$ and $i = \sqrt{-1}$, i.e. to decouple them from the primary binding.

Exercise 15.7. Try to detect a pattern in the determinant formula (*Leibniz*3).

Eigenmath returns the (*Leibniz*3) formula in an evaluated and automatically simplified shape. Therefore a possible structure is hidden behind this long term, which we try to shed some light on. So let's go back to the 90's of the last century and look at a screenshot of the ancient CAS Derive ^{forWindows} on the same theme:



We know, that the denominator in command line #18 must be the determinant of the system matrix M. We conclude, that the (*Leibniz*3) formula should be

$$det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \overset{Leibniz3}{=} aei - afh - bdi + bfg + cdh - ceg \tag{15.1}$$

$$\overset{\#18}{=} a \cdot (ei - fh) + b \cdot (fg - di) + c \cdot (dh - eg) \tag{15.2}$$

$$\overset{Leibniz2}{=} a \cdot det\begin{bmatrix} e & f \\ h & i \end{bmatrix} - b \cdot det\begin{bmatrix} d & f \\ g & i \end{bmatrix} + c \cdot det\begin{bmatrix} d & e \\ g & h \end{bmatrix} \tag{15.3}$$

Equation (15.3) shows the structure, we are looking for. This structured formula for the calculation of the 3×3 determinant is known as the LAPLACE *expansion of the determinant* and is good to memorize and easily generalizable.

Exercise 15.8. Argue, why we introduce the '-' sign in the middle term of equation (15.3). Argue, why the explicit EIGENMATH term (15.1) seems to come from in intern simplification of (15.3). Why can formula (15.1) be memorized with this knowledge in mind?

Example.

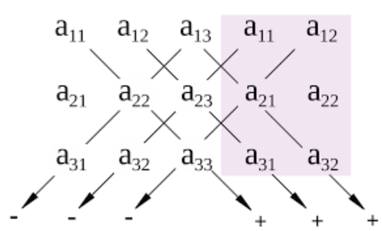
$$det \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \overset{(15.3)}{=} 1 \cdot det \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix} - 2 \cdot det \begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix} + 3 \cdot det \begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix} \quad (15.4)$$

$$\overset{Leibniz2}{=} 1 \cdot (-3) - 2 \cdot (-6) + 3 \cdot (-3) \quad (15.5)$$

$$= 0 \quad (15.6)$$

Exercise 15.9. Here is the so-called rule of *Sarrus*[7], which is a pattern to memorize the calculation of the determinant along the (*Leibniz*3) formula (15.1), explain:

$$det \begin{bmatrix} 2 & 3 & 5 \\ -1 & 4 & 6 \\ 3 & -2 & 7 \end{bmatrix} =$$



$$= 2 \cdot 4 \cdot 7 + \cdots - (3 \cdot (-1) \cdot 7) = 105$$

Check the calculation with ▷EIGENMATH.

15.3 LAPLACIAN expansion – the nD CRAMER rule

With the insight gained into the apparatus of the determinant formulas in its different characteristics of the LEIBNIZ sum or LAPLACE expansion, we are now able to formulate general rules as well as for the CRAMER rule as for the calculation of determinants.

15.3.1 The nD CRAMER rule

Because our EIGENMATH function `Replace(..)` works independent of the dimension of the matrix, we only have to generalize the CRAMER rule implementation and orientate us at 15.2.2 code line #9. We have the lexicon DERIVE : `vector(.., i,1,DIMENSION(a))` EIGENM: `for(i,1,dim(A,1), ..)`

```
## CRAMER rule for n x n matrices
Cramer(A,B) = do( n = dim(A,1),        --(1)
                  Z = zero(2,n),       --(2)
                  Y = Z[1],            --(3)
```

[7] we borrow the a_{ij} pattern from `https://wiki2.org/en/Determinant`

```
                    for( i,1,n,          --(4)
                         Y[i] = det(Replace(B,A,i))/det(A) ),
                    Y )                  --(5)

A = ((3,2,-1),(2,1,2),(0,0,1))
B = (6,4,1)

cNN = Cramer(A,B)    --(6)
cNN

Cramer(A,B)[2]       --(7)
```

Try it: ▷*Click here to run the script.*

Comment. In (1) we say, that the `for`–loop will use all n columns of matrix A. (2) installs a $2\times n$ container matrix Z, who is initially filled with zeros. But we use only the first row $Z[1]$ of it to save the calculated solution components in variable Y. The calculation is done in (4) and the full result is returned in (5).

Exercise 15.10. Use CRAMER(..) to solve Ex.15.9.

15.3.2 The LAPLACIAN expansion of a 3×3 determinant

We recall the structure formula (15.3) to compute a 3×3 determinant, the so-called LAPLACE expansion:

$$det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \overset{Leibniz3}{=} a\cdot det\begin{bmatrix} e & f \\ h & i \end{bmatrix} - b\cdot det\begin{bmatrix} d & f \\ g & i \end{bmatrix} + c\cdot det\begin{bmatrix} d & e \\ g & h \end{bmatrix} \tag{15.7}$$

Each of the 1D–smaller 2×2 determinants on the RHS of (15.8) is called a *minor* of the original matrix on the LHS. It is this pattern, which can easily be extended to a general procedure (e.g. a definition) to calculate the determinant of a matrix, know as its LAPLACE *expansion*. Let's first formulate the LAPLACE equation (15.7) using EIGENMATH:

```
A = ((a,b,c),(e,f,g),(h,i,j))

detLap(A) = A[1,1] * minor(A,1,1) -
            A[1,2] * minor(A,1,2) +
            A[1,3] * minor(A,1,3)

detLap(A)
```

The EIGENMATH output shows the expected value: $aei - afh - bdi + bfg + cdh - ceg$ Here we show the full session and comment a bit about it:

```
do( e=quote(e), i=quote(i))

A = ((a,b,c),(e,f,g),(h,i,j))
A


det(A)

minormatrix(A,1,1)              --(1)
minormatrix(A,1,2)

minor(A,1,1)                    --(2)
minor(A,1,2)

detA = A[1,1]*minor(A,1,1) -    --(3)
       A[1,2]*minor(A,1,2) +
       A[1,3]*minor(A,1,3)
detA

det(A) == detA                  --(4)
```

$$A = \begin{bmatrix} a & b & c \\ e & f & g \\ h & i & j \end{bmatrix}$$

$$afj - agi - bej + bgh + cei - cfh$$

$$\begin{bmatrix} f & g \\ i & j \end{bmatrix}$$

$$\begin{bmatrix} e & g \\ h & j \end{bmatrix}$$

$$fj - gi$$

$$ej - gh$$

$$d_{etA} = afj - agi - bej + bgh + cei - cfh$$

$$1$$

Comment. We define a general 3×3 matrix A, calculate its *det*erminant using EIGENMATHs build-in function `det(..)` and pick out two scaled down quadratic sub-matrices of A using EIGENMATHs build-in function `minormatrix(..)`, see (1). The corresponding `minor(A,i,j)` is defined to be the determinant of this 2×2–submatrix that results from A by removing the i–th row and the j–th column, see (2). In (3) we compute the RHS of formula (15.7) and check in (4) with success ("1"), that both terms are equal, i.e. $LHS = RHS$. Formula (3) was abstracted above to an explicit executable function.
Try it: ▷*Click here to run the test.*

15.3.3 The LAPLACIAN expansion of a general $n \times n$ determinant

In 15.3.1 we gave the general version of the CRAMER rule in EIGENMATH. We now show the EIGENMATH implementation of the LAPLACIAN expansion of the determinant (along the elements of the i-th row) of a general $n \times n$ matrix. This is a straight forward translation of the corresponding mathematical formula.

Math: for $A = (a_{i,j})_{\substack{i=1..n \\ j=1..n}}$ we have with $M_{ij} =$ `minor(A,i,j)`

$$\det(A, i) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij}$$

EIGENMATH:

```
detLaplace(A,i) = sum(j,1,dim(A,1), (-1)^(i+j)*A[i,j]*minor(A,i,j) )
```

```
A = ((3,2,-1),(2,1,2),(0,0,1))

detLaplace(A,1)
detLaplace(A,2)
```

Try it: ▷*Click here to run the script.*
• *Remember*: The LAPLACE expansion expresses the determinant of a matrix in terms of its minors. The *minor* $M_{i,j}$ =`minor(A,i,j)` is defined to be the determinant of the $(n-1)\times(n-1)$ submatrix that results from A by removing the i–th row and the j–th column. The signed expression $(-1)^{i+j}M_{i,j}$ is known as a *cofactor*, in EIGENMATH language more precise as `cofactor(A,i,j)`.
• If we are only interested in the calculation of *det* without the possible choice of a suitable row i to expand along, we may fix the first row and write:

```
deti(A) = sum(j,1,dim(A,1), (-1)^(1+j)*A[1,j]*minor(A,1,j) )
```

• Here is the LAPLACE expansion in its compact cofactor version:

```
detCofactor(A,i) = sum(j,1,dim(A,1), A[i,j]*cofactor(A,i,j) )
```

15.4 Concept net with the Adjugate

Now we understand the systematic process of the recursive determinant computation via the LAPLACE expansionand want to reflect again the calculation of the solution vector X of an linear system of the form $A * X = B$ using CRAMER rule with the new insights. According to the CRAMER rule, certain substitutions were made to construct the numerator determinant of the solution vector. Let's look at this first.
For this exploration a new mathematical concept is appropriate: the so-called `adj`*ugate*. This concept will lead to a coherent network and distillates out the core part of the concepts inverse or the CRAMER rule.

15.4.1 Exploring the Adjugate

We do the following exploration in shape of a socratic dialog with our CAS EIGENMATH and use the method of pattern matching. We start with *a question to* EIGENMATH*: what is the" *adjugate*" of a general matrix A*?

```
# Adjugate = adj
A = ((a,b,c),(d,e,f),(g,h,i))
adj(A)
```

EIGENMATH answer:

$$\texttt{adj(A)} = \begin{bmatrix} ei - fh & -bi + ch & bf - ce \\ -di + fg & ai - cg & -af + cd \\ dh - eg & -ah + bg & ae - bd \end{bmatrix}$$

The entries inside the 3×3 adj − matrix reminds of the Laplace expansion of the determinant of A. We go a step backwards and construct a helper function $MiMa$, which returns all 9 minormatrixes of A.

```
#  helper function to output all minormatrix'es
MiMa(A)= do( e = quote(e), i = quote(i),    --(1)
             MM = zero(3,3,2,2),            --(2)
             for(k,1,3,
             for(j,1,3,
                 MM[k,j] = minormatrix(A,k,j))),  --(3)
             MM)                            --(4)

A = ((a,b,c),(d,e,f),(g,h,i))
MiMa(A)
```

▷ *Click here to run the script.*
Eigenmath answer:

$$\texttt{MiMa(A)} = \begin{bmatrix} \begin{bmatrix} e & f \\ h & i \end{bmatrix} & \begin{bmatrix} d & f \\ g & i \end{bmatrix} & \begin{bmatrix} d & e \\ g & h \end{bmatrix} \\ \begin{bmatrix} b & c \\ h & i \end{bmatrix} & \begin{bmatrix} a & c \\ g & i \end{bmatrix} & \begin{bmatrix} a & b \\ g & h \end{bmatrix} \\ \begin{bmatrix} b & c \\ e & f \end{bmatrix} & \begin{bmatrix} a & c \\ d & f \end{bmatrix} & \begin{bmatrix} a & b \\ d & e \end{bmatrix} \end{bmatrix}$$

Comment. Here are some comments about the implementation of MiMa, a pre–version of adj. Code line (1) detaches the identifiers e and i from being $e = exp(1)$ and $i = \sqrt{-1}$. In (2) we define a 'tensor' MM as a 3×3 matrix, whose entries are itself 2×2 matrices. The for–loop in 3 fills alls 9 elements of tensor MM with the minormatrix'es. In (4) we return the tensor matrixf MM.

What do we observe?

1. The 1st entry adj(A)[1,1] equals the determinant of the 1st entry matrix of MiMa(A)[1,1], i.e. $adj(A)[1,1] = \det(\text{MiMa}(A)[1,1])$.
2. But: The 1st row adj(A)[1] matches the 1st column of MiMa(A).
3. Therefore: the transpose of MiMa(A) matches in pattern with adj(A).
4. Why? Wait, see below. You can only understand the *adjugate* in retrospect.

We now look back at the nominator of the 3×3 CRAMER rule, see e.g. 15.2.2 #18.

```
    Replace((p,q,r),A,1)                   --(1)
det( Replace((p,q,r),A,1) )                --(2)
p*(e*i-f*h) + q*(c*h-b*i) + r*(b*f-c*e)    --(3)

p * det(((e,f),(h,i))) -                   --(4)
q * det(((b,c),(h,i))) +
r * det(((b,c),(e,f)))

p * cofactor(A,1,1) +                      --(5)
q * cofactor(A,2,1) +
r * cofactor(A,3,1)
```

$$\begin{bmatrix} p & b & c \\ q & e & f \\ r & h & i \end{bmatrix}$$

$$bfr - biq - cer + chq + eip - fhp$$

$$bfr - biq - cer + chq + eip - fhp$$

$$bfr - biq - cer + chq + eip - fhp$$

$$bfr - biq - cer + chq + eip - fhp$$

▷ *Click here to run the script.*
What do we observe here?

1. Formulas (2) .. (5) give back the same term, i.e. they are equivalent.
2. All 4 terms give back the x–coordinate of the solution vector X using CRAMER rule.
3. (2) resp. (3) corresponds to the 1st *row* of `adj(A)`.
   It is like a linear combination of these entries with the factors p, q, r.
4. (4) calculates the x–value of the solution vector in the CRAMER rule using the determinants of the 1st *column* of the `minormatrix` tensor `MiMa`.
5. (5) encodes expression (4) using the *cofactor* abbreviation and *allows to forget about the minus sign* in (4).

Exercise 15.11. Give analogous formulas, if [p, q, r] replaces the 2nd or 3rd column of A.

Exercise 15.12. Repeat the exploration above using the concrete linear system$A * X = B$ e.g.

$$\begin{bmatrix} 2 & 2 & 4 \\ 1 & 2 & -5 \\ 3 & 1 & -3 \end{bmatrix} * \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 5 \end{bmatrix}$$

Exercise 15.13. For matrix $A = \begin{bmatrix} 2 & 2 & 4 \\ 1 & 2 & -5 \\ 3 & 1 & -3 \end{bmatrix}$, what is using paper'n pencil

- ∘ cofactor$(A, 3, 1)$
- ∘ adj$(A)[1] * [5, 4, 5]$
- ∘ adj$(A) * B$

Do this exercise with EIGENMATH.

15.4.2 The CRAMER rule and the Adjugate

How is the CRAMER rule formula hidden in the *adjugate*[8]?
How can one calculate the solution X of a linear system $A * X = B$ with the help of the adjugate of the system matrix A?

[8] alias "accompanying matrix", "adjunct"

So we explore how we can re-formulate the CRAMER rule formula using the adjugate `adj` and dive into a EIGENMATH session:

```
A = ((2,2,4),(1,2,-5),(3,1,-3))
B = (5,4,5)

CoMa(A) = do( e = quote(e), i = quote(i),
              MM = zero(3,3),
              for(k,1,3,
              for(j,1,3,
                 MM[k,j] = cofactor(A,k,j))),
              MM)                --(1)

cA = CoMa(A)                     --(2)
cA

aA = adj(A)                      --(3)
aA

           CoMa(A) == adj(A)     -- 0 = No
transpose(CoMa(A)) == adj(A)     -- 1 = Yes      (4)

# Term (5) of last session is nominator of x-value
# = (1st column of CoMa(A)) * B
# = (1st row of adj(A)) * B
#   i.e. in EIGENMATH:

dot(transpose(CoMa(A))[1], B)                  --(5)
dot( adj(A)[1], B)

# In summa we have:
-----------------------------------------------------
CramerAdj(A,B)= dot(adj(A),B)/det(A)
-----------------------------------------------------

CramerAdj(A,B)                       --result: X=(55/46,...)
```

EIGENMATH output:

$$c_a = \begin{bmatrix} -1 & -12 & -5 \\ 10 & -18 & 4 \\ -18 & 14 & 2 \end{bmatrix} \quad a_A = \begin{bmatrix} -1 & 10 & -18 \\ -12 & -18 & 14 \\ -5 & 4 & 2 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{55}{46} \\ \frac{31}{23} \\ -\frac{1}{46} \end{bmatrix}$$

▷ *Click here to run the session.*

Comment. `CoMa` is the matrix, whose elements are the (i, j)–*cofactors* of A, i.e. the signed *determinants* of the 2×2 submatrices ('*minormatrix*'), that results from A by removing the i–th row and the j–th column. Comparing (2) and (3) we see, that *the adjugate is the transpose of the cofactor–matrix*. In (5) we compute the x-value of the solution vector X two way, both returning -55. Then we define the CRAMER rule using the adjugate.

As a result, we get the following *dimension-independent solution* CRAMER *rule formula for uniquely solvable linear systems* of equations $A * X = B$:

Math	EIGENMATH
$X = \frac{adj(A) \bullet B}{det(A)}$	`Cramer(A,B) = dot(adj(A),B)/det(A)`

Remark. With this dimension-independent solution CRAMER rule formula for uniquely solvable linear systems of equations $A * X = B$ we do not need the mental helper function `Replace` any more, which was useful for paper'n pencilcalculations using the CRAMER rule.

Exercise 15.14. Compute the y, z–components of the linear system of 15.4.2 analog to code line (5) using first paper'n pencil and then EIGENMATH.

15.4.3 The inverse A^{-1} and the Adjugate

Question: if the solution of a linear system with the inverse of A can be calculated via the CRAMER rule also with the adjugate, what is then the relationship between the inverse of A and the adjugate of A, i.e. $A^{-1} \overset{?}{\sim} adj(A)$?
In the CRAMER rule process, the formation of the inverse of the matrix must be implicitly hidden. In conclusion, we want to explore this and detect a famous connection between the determinant $det(A)$, the adjugate $adj(A)$ and the inverse A^{-1} of a given matrix A.
Enjoy the following short reflection in EIGENMATH. Without words.

```
A=((2,2,4),(1,2,-5),(3,1,-3))

dot(A, inv(A))              -- A*A^(-1)

dot(A, adj(A))              -- A*adj(A)

det(A)

dot(A, adj(A))/det(A)       -- (A*adj(A))/det(A)

------------------------ inv-adj-det relation:
invAdj(M)= adj(M)/det(M)
------------------------

invAdj(A)
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -46 & 0 & 0 \\ 0 & -46 & 0 \\ 0 & 0 & -46 \end{bmatrix}$$

$$-46$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{46} & -\frac{5}{23} & \frac{9}{23} \\ \frac{6}{23} & \frac{9}{23} & -\frac{7}{23} \\ \frac{5}{46} & -\frac{2}{23} & -\frac{1}{23} \end{bmatrix}$$

▷*Click here to run the session.*

Exercise 15.15. Comment exploration 15.4.3 in your own words.

Exercise 15.16. In which line of the session you are able to conclude that $\frac{adj(A)}{det(A)}$ is the inverse of A. Why?.

Exercise 15.17. Verify each of the first 4 code line through a calculation by paper'n pencil.

Fact: For any invertible square matrix A

Math	EIGENMATH
$A^{-1} = \frac{adj(A)}{det(A)}$	`inv(A) == adj(A)/det(A)`

Remark. The construction of the inverse matrix for a given matrix A is now divided in its main parts: *the inverse of matrix A is the quotient of the adjugate of A and its determinant.* In particular it can be seen that the determinant plays a decisive role for the existence of the inverse matrix. *Memorize*:

- The *adjugate* of a matrix is the transpose of its cofactor matrix.
- The *inverse* of a matrix is the quotient of its adjugate and its determinant.
- The adjugate of a matrix is the transpose of its cofactor matrix.

Exercise 15.18. The CRAMER rule says:

$$X = \frac{adj(A) \bullet B}{det(A)} = \frac{adj(A)}{det(A)} \bullet B = A^{-1} * B$$

What do you answer, if someone argue, we had gone into a circular reasoning[9], because we know

$$A * X = B \rightsquigarrow X = A^{-1} * B$$

Exercise 15.19. In your own words: what is the adjugate of a matrix useful and valuable for?

15.4.4 Concept map:

Matrix, *minorMatrix*, cofactorMatrix, Adjugate, Inverse

$$\begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \rightsquigarrow \begin{bmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{bmatrix} \rightsquigarrow \begin{bmatrix} \det\square & \det\square & \det\square \\ \det\square & \det\square & \det\square \\ \det\square & \det\square & \det\square \end{bmatrix} \xrightarrow{\square^t} adj(\square) \longrightarrow \square^{-1}$$

$\downarrow$

`Cramer(A,B)`

- Explain for yourself. Think about it. Watch: $\square^t$!

[9] https://en.wikipedia.org/wiki/Circular_reasoning

Problems.

P140. Multi–variant solution of a 3×3 linear system
Consider the following system of linear 3x3 equations in matrix form A.X = B given by:

$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & -3 & 2 \\ 0 & 1 & 6 \end{bmatrix} * \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}$$

a. Solve the linear system with `RREF(..)` function.
b. Just determine the x-coordinate using the CRAMER rule function.
c. Determine the complete solution vector [x, y, z] using the CRAMER rule function.
d. Solve the LS by multiplying by the inverse A^{-1}.
∘ What is the advantage of CRAMER rule compared to a solution using A^{-1} or `RREF()`?

P141. Parameter dependency
For which values of k does the following 3x3-LGS have a unique solution?

$$\begin{bmatrix} kx + y + z = 2 \\ x + ky + z = 3 \\ x + y + kz = 4 \end{bmatrix}$$

P142. Quadratic fitting.
If one wants to lay a parabola $y = a_0 + a_1 x + a_2^2$ through three points $P_1[x_1, y_1], P_2[x_2, y_2]$ and $P_3[x_3, y_3]$, one looks for a solution $[a_0, a_1, a_2]$ of the following three linear equations

$$\begin{aligned} a_0 + a_1 x_1 + a_2 x_1^2 &= y_1 \\ a_0 + a_1 x_2 + a_2 x_2^2 &= y_2 \\ a_0 + a_1 x_3 + a_2 x_3^2 &= y_3 \end{aligned}$$

a. Explain that a solution exists if and only if

$$det \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix} \neq 0$$

b. Which parabola goes through [2,1] and [5, -1] and [-1,0]?
Try drawing the parabola you are looking for with CALCPLOT3D.

P143. Straight line equation via *det*.
a. Show with/without EIGENMATH that the straight line ℓ_{PQ} through the points $P(p_1, p_2)$ and $Q(q_1, q_2)$ is determined through the following equation:

$$det \begin{bmatrix} 1 & x & y \\ 1 & p_1 & q_1 \\ 1 & p_2 & q_2 \end{bmatrix} = 0$$

b. According to a) give the straight line through $P(2,1)$ and $Q(4,5)$.
c. Make a quality plot of ℓ with CalcPlot3D.
d. Calculate the equation of ℓ alternatively using the point-slope approach or the two-point approach.

P144. Lines of adjugate.
Consider the following fragment of an Eigenmath script:

```
A=((2,2,4),(1,2,-5),(3,1,-3))
dot( adj(A)[1], (5,4,5)) /det(A)
```

First formulate an assumption in words what is to be calculated, then do the math by hand and finally check the result with Eigenmath. What is the result of the code snippet?

P145. Chessboard rule.
When forming the adjugate of a matrix one must carefully pay attention to the signs of the sub-determinants, i.e. the minors. The sign of the minor changes according to the following easily noticeable pattern:

$$\begin{bmatrix} +1 & -1 & +1 \\ -1 & +1 & -1 \\ +1 & -1 & +1 \end{bmatrix}$$

a. Explain.
b. To what extent is this sign pattern already taken into account in cofactor expansion?
c. The formula for the inverse

$$A^{-1} = \frac{1}{det(A)} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

of a 2×2 matrix A was determined by solving a 2×2 linear system.
Derive this formula as a special case of $A^{-1} = \frac{adj(A)}{det(A)}$ observing the chessboard pattern.

P146. Determinant definition.
Discuss whether you can compute the *det* function for a matrix M also through the term

$$DET(M) = adj(M)_1 \bullet M_1$$

Study examples. Explain and defend your opinion. Write `DET` in Eigenmath's programming language.

P147. Relationship between determinates, adjuncts and inverses
Consider the following 2×2 or 3×3 matrices:

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} B = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} U = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix} V = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} W = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 2 & 4 \\ 1 & 1 & 2 \end{bmatrix}$$

a. Calculate the determinants of A, B, U, V and W.
b. Which of the matrices A, B, U, V and W has an inverse?
c. Calculate the adjjugates of A, B, U, V and W, those of A, B and U also by hand.
d. Calculate the matrix products in each case

1. $A \bullet adj(A) =$

2. $B \bullet adj(B) =$

3. $U \bullet adj(U) =$

4. $W \bullet adj(W) =$

○ Compare each with the corresponding inverse!

P148. Area of Triangle and Dependency of vectors.
For three points $A = [a_1, a_2]$, $B = [b_1, b_2]$ and $C = [c_1, c_2]$ in the plane we consider the EIGENMATH function

```
F(A,B,C) = det( ((1,A[1],A[2]), (1,B[1],B[2]), (1,C[1],C[2]) ))
```

a. Calculate `F((0,0), (4,1), (3,3))` using EIGENMATH and also paper'n pencil.
b. Draw the points as a triangle (polygon) with CALCPLOT3D.
c. Show that $F(A, B, C)$ gives the *signed area* of the triangle $\triangle ABC$.
d. How can one interpret the sign?
e. If $F(A, B, C) \neq 0$, then the three points (A, B, C) are called *affine-independent* alias the two vectors $(A - C, B - C) = (\overrightarrow{CA}, \overrightarrow{CB})$ *linear–independent*,
in case $F(A, B, C) = 0$ they are called dependent. Explain.
▷ *Click here to run the function.*

15.5 Adjugate, Cross, Box and Cramer

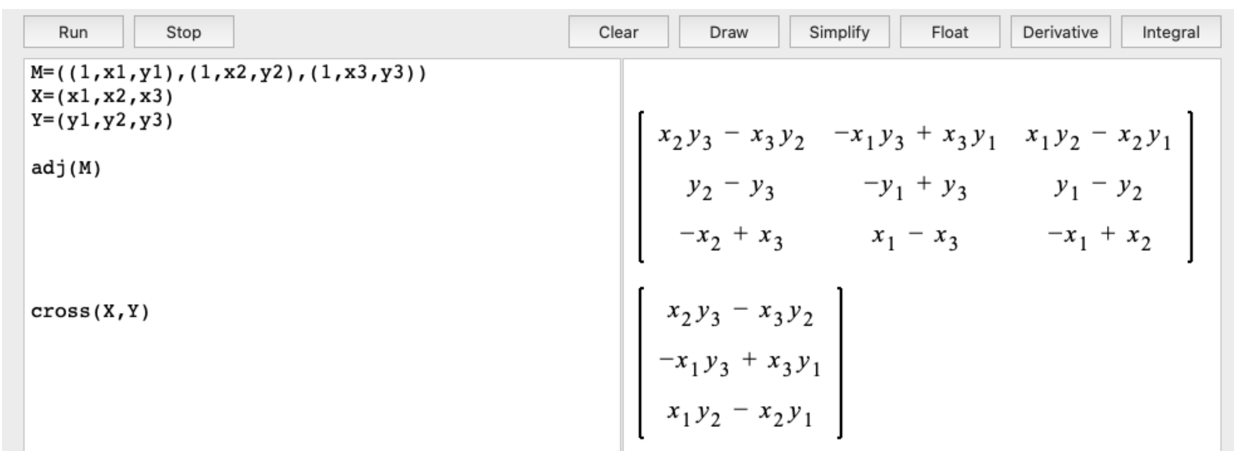
The following considerations and results apply specifically to 3×3 matrices, 3×3 linear systems and 3×3 determinants. This is why the names of functions often have a 3 at the end as a reminder of this restriction, e.g. `DET3`. We demonstrate, how to use the adjugate to define the *cross* product and and then how to calculate the determinant using cross. Finally we interpret the solution components of the 3D Cramer rule as ratios of volumina.

15.5.1 The Adjugate and the Crossproduct

To detect a hidden connection between the *vectorproduct* alias *crossproduct* and the adjugate we start a short Eigenmath exploration. We consider a "special general" 3×3 matrix and its adjugate to get general results. We study the following matrix

$$M = \begin{bmatrix} 1 & x1 & y1 \\ 1 & x2 & y2 \\ 1 & x3 & y3 \end{bmatrix}$$

This matrix M is *special*, because its first column consists only of "1"s. It is a bit *general*, because the other columns have variable elements. Now we have



▷ *Click here to run the exploration.*
What do we recognize? We know that the elements of the adjugate $adj(A)$ of a matrix A are the determinants of its smaller submatrices, its 'minormatrices'. Therefore the first row of the adj(ugate) remembers at the Leibniz formula for 2×2 determinants, while at the same time the entries of the *cross*product vector have the same values. Therefore we conclude: $cross(X,Y) == adj(M)[1]$ and we can define a procedure to calculate *cross* using the *adj*:

```
CROSS(X,Y) = do( M=zero(3,3),       --(1)
                 M[1]=(1,1,1),      --(2)
                 M[2]=X,            --(3)
                 M[3]=Y,            --(4)
```

```
                    Mt=transpose(M),   --(5)
                    adj(Mt)[1])        --(6)

     CROSS(X,Y)
```

▷*Click here to run the exploration.*

Comment. This cross product function $CROSS$ via the *adj*(ugate) reflects 1:1 the mental or paper'n pencil calculation of the cross product and presents a step-by-step recipe:

1. Take an 'empty' matrix M.
2. Fill its first column with 1s.
3. Fill its second column with the first vector entry X for cross.
4. Fill its third column with the second vector entry Y for cross.
5. Take the first row of $adj(M)$, i.e. take the *cofactors* of the first column of M as entries for the cross product. In other words: *the first line of the adjugate of this matrix equals the cross product.*

In summa:

$$cross(\begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix}, \begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix}) = adj(\begin{bmatrix} 1 & x1 & y1 \\ 1 & x2 & y2 \\ 1 & x3 & y3 \end{bmatrix})_{1,\bullet}$$

or as mental concept

$$\overset{\text{Matrix}}{\begin{bmatrix} 1 & \cdot & \cdot \\ 1 & \cdot & \cdot \\ 1 & \cdot & \cdot \end{bmatrix}} \quad \overset{\text{Adjugate}}{\longrightarrow adj(\Box) \longrightarrow} \quad \overset{\text{cross}}{adj(\Box)_{1.Row}} = cross(\vdots|\vdots)$$

High time for an

Exercise 15.20. We adopt the mathematical operator symbol ... × .. for the cross product, i.e. we write shortly X×Y $\overset{def}{=}$ $cross(X, Y)$. Now determine the following cross products using 1. paper'n pencil 2. EIGENMATH's bulid-in `cross` and 3. our user–defined `CROSS` function.

a. $[1, 2, 3] \times [4, 5, 6]$
b. $[1, 2, 3] \times [-3, 6, -3]$
c. $[1, 2, 3] \times [1, 2, 3]$
d. Do a free training with self chosen vector pairs until you feel competent.
▷*Please invoke* EIGENMATH

15.5.2 The 3×3 determinant via a cross product

In the following exploration we use the predefined EIGENMATH function `cross(..)` to calculate cross products. Consider a 3×3 matrix M and his row vectors A, B, C thought as individual objects. Then we have:

```
##### det via cross
M=((a1,a2,a3),(b1,b2,b3),(c1,c2,c3))

A=(a1,a2,a3)
B=(b1,b2,b3)
C=(c1,c2,c3)

M[1]
M[2]
M[3]

det(M)              --(1)
dot(A,cross(B,C))   --(2)   A*(BxC)
adj(M)[1]           --(3)

----------------------------------- (4)

DET3(M)= dot(M[1],cross(M[2],M[3]))

-----------------------------------

DET3(M)             --(4)
DET3((A,B,C))       --(5)
```

The tree different invocations (1), (2) and (3) returns the same term for the calculation of the determinant of M:

$$a_1 b_2 c_3 - a_1 b_3 c_2 - a_2 b_1 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1$$

▷ *Click here to run the script.*

Therefore we have the possibility to define the calculation function–term for a 3×3 *det* (erminant) via formula (4). The advantage is, that we are able to input the matrix as 3 individual vectors, see (4) and (5). In some geometrical situation in $\mathbb{R}^3$ this is comfortable, look at the problems.

From this observation we have also another easy-to-remember calculation option for the value of a 3×3 determinant if the rows/columns of the associated matrix are interpreted as individual vectors.

	Math	EIGENMATH
DET3(A,B,C)=	$A * (B \times C)$	`dot(A, cross(B,C))`

Exercise 15.21. Calculate the determinant of $M = ((2,2,4),(1,2,-5),(3,1,-3))$ using function $DET3$. Verify your result with the build–in function det.

15.5.3 A geometric interpretation of the 3×3 Cramer rule as ratio of volumes

Looking back at §9.6 we see that

$$DET3(a,b,c) = \frac{\textit{Math: } a * (b \times c)}{\text{EIGENM: } \texttt{dot(a,cross(b,c))}} = Box(a,b,c)$$

Therefore, analogous to the interpretation of a 2×2 determinant as the area of a parallelogram with the columns of the matrix as edges, the value of a 3×3 determinant can be interpreted as the volume of an parallelepiped ("feldspar") with the columns of the matrix as edges. We have the exploration:

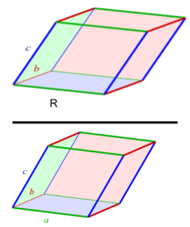
```
# EXAMPLE 3x3 LINEAR SYSTEM
# 2x+2y+4z=5
# 1x+2y-5z=4
# 3x+1y-3z=5

A = (2, 2,  4)    --LHS
B = (1, 2, -5)
C = (3, 1, -3)
R = (5, 4,  5)    --RHS
R

M = (A,B,C)
M

Box(A,B,C)        --(a)
det((A,B,C))      --(b)
DET3((A,B,C))     --(c)

# x coordinate of solution vector X=(x,y,z) via CRAMER with Box()
Cramer3x(M,R)= do( M=transpose(M),          --(3)
                   Box(   R, M[2], M[3]) /  --(4)
                   Box(M[1], M[2], M[3]))   --(5)


Cramer3x( ((A,B,C)), R)                     --(6)
Cramer3x(M,R)                               --(7)
```

Eigenmath output: (a),(b),(c) equals -46 and (6),(7) equals $\frac{55}{46}$

Comment. Eigenmath is row oriented. Therefore we have to plug in the RHS R and the rows of system matrix M as columns, i.e. we have to transpose our data in code line (3).

Line (4) shows the replacement of the RHS R for the 1st column of the system matrix for calculating the nominator. Line (5) is the constant system matrix M. Because we use function Box() in the implementation of the new CRAMER rule formula (3), *the solution x is interpretable as ratio of the two volumes* (Boxes) in the nominator and the denominator. What a wonderful result. And so simple to memorize. That's math ♡[10]

`Cramer3x( ((a,b,c)), R) =`



▷ *Click here to run the script.*

Exercise 15.22. Write analogous CRAMER rule formulas `Cramer3y( ((a,b,c)), R)` and `Cramer3z( ((a,b,c)), R)` for the computation of the y and z coordinate of the solution vector $X = (x, y, z)$. Give an implementation of the 'full' CRAMER rule `Cramer3( ((a,b,c)), R)`, which returns the complete solution vector $X = (x, y, z)$ of the linear system using the partiell CRAMER rule formulas `Cramer3x`, `Cramer3y` and `Cramer3z`.
Test your formulas using the linear system of the exploration.
Check your result using the inverse M^{-1} of the system matrix.

Problems.

P149. Multi-variant solution of a 3×3 linear system.
Consider the following system of 3 linear equationsin 3 unkowns in matrix form $A * X = B$ given by

$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & -3 & 2 \\ 0 & 1 & 6 \end{bmatrix} * \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 3 \end{bmatrix}$$

a. Solve the linear system using the mental CRAMER rule formula by paper'n pencil.
b. Determine *only* the x–coordinate of solution vector $X = [x, y, z]$.
c. Determine the complete solution vector $[x, y, z]$ using your `Cramer3` function.
d. Verify your solution by $X = A^{-1} * B$ using EIGENMATH as your assistent.

P150. CRAMER **ruleusing `Box`.**
Consider this 3×3 linear system:

$$\begin{bmatrix} 1 \cdot x + 2 \cdot y + 3 \cdot z = 1 \\ 4 \cdot x + 5 \cdot y + 6 \cdot z = 0 \\ 7 \cdot x + 7 \cdot y + 14 \cdot z = 21 \end{bmatrix}$$

a. Write the linear system in matrix shape $A * X = B$.

[10] The picture of the parallelepiped in the denominator is from `https://de.wikipedia.org/wiki/Datei:Parallelepiped-0.svg`

b. Determine the *nominator* of the x–coordinate of the solution using `Box(..)`.
c. Compute the y component of the solution by only using `Box` or `DET3` function.
c. Compute the z component of the solution by only using `cross` and `dot` function.
d. Verify your solution by $X = A^{-1} * B$ using EIGENMATH as your assistent.

P151. Algebraic properties of `cross` alias ×.
In manual calculations and theoretical considerations, one writes the *cross* product function `cross(X, Y)` usually as 2-ary operator $X \times Y$.

a. Calculate by hand, check with EIGENMATH:

$[1, 2, 3] \times [4, 5, 6], [1, 2, 3] \times [1, 2, 3], [4, 5, 6] \times [1, 2, 3], [1, 2, 3] \times [4, 5, 6]$

b. Show that the name *cross* **product** is justified because the following rules apply: the cross product is (for $A, B, C \in \mathbb{R}^3$)

1. *homogeneous*: $A \times (k \cdot B) = k \cdot (A \times B)$ with $k \in \mathbb{R}$
2. *distributive*: $A \times (B + C) = (A \times B) + (A \times C)$
3. (In contrast to the normal multiplication of numbers, however, :) *anticommutative*: $A \times B = -(B \times A)$
4. *alternating*: $A \times A = \vec{0}$ (always, even if A is not a zero vector!)

c. Verify the above rules using EIGENMATH and arbitrary 'general' vectors, e.g. $A = (a1, a2, a3)$ etc. Search for further properties/rules for ×! Prove with EIGENMATH.

c. Compare the *cross* product × in $\mathbb{R}^3$ with the *wedge* product in $\mathbb{R}^2$; look for similarities and note the differences.

Note: the result of the cross product is a vector. Hence it is often also called 'the' *vector* product in $\mathbb{R}^3$.

P152. Algebraic rules for the *Box* product of three vectors in space.
The *Box* product is sometimes called the 3D *wedge* product or *cap* product for vectors in space $\mathbb{R}^3$ and is then denoted as $a \wedge b \wedge c$ instead of $Box(a, b, c)$.
In analogy to the previous exercise, search for algebraic rules for the $Box \equiv wedge$ product.

Remark. 1. The function value `Box(a,b,c)` alias the value of the *wedge* operator $a \wedge b \wedge c$ is a real number, that corresponds to the value of a 3×3 determinant; however, the 3 inputs of the 3D wedge product are viewed three individual vectors and not as components of one 3×3 matrix as in *det*.
2. We have declared a function `DET3(M) = dot(M[1],cross(M[2],M[3]))`, which operates on 3×3 matrices M. We may also define `Det3(a,b,c)= dot(a,cross(b,c))`, which awaits 3 vectors as input. $Det3$ would then be the same as Box and would be a so-called *multilinear function*.
a. $[1, 2, 3] \wedge [4, 5, 6] \wedge [1, 2, 3] =?$
b. $Det3([4, 5, 6], [1, 2, 3], [-1, 0, 1] =?$

P153. Orthogonal vectors.
a. Draw the two vectors [4,1] and [-1,4] in the Cartesian coordinate system of CalcPlot3D. Calculate their scalar product.
b. Form the dot product of the vectors [4,1,3] and [-3,0,4].
○ *Define*: Two vectors X and Y are called *perpendicular* or *orthogonal* if their scalar product is zero, i.e

$$X \perp Y \overset{def}{\Longleftrightarrow} X \bullet Y = 0 \qquad (\text{in EIGENMATH}: dot(X,Y) = 0)$$

c. Justify with Eigenmath: $X \perp (Y \times Y)$ and $Y \perp (X \times Y)$
d. Interpret geometrically: $Det3(A,B,C) = 0$.

P154. The *trace* of a matrix and Cayley's theorem)
Define the so-called *trace* of a matrix M to be the sum of its diagonal elements, e.g. in the 2D case:

$$trace \begin{bmatrix} a & b \\ c & d \end{bmatrix} \overset{def}{=} a + d \overset{EigenM}{=} \texttt{contract}(((\texttt{a},\texttt{b}),(\texttt{c},\texttt{d})))$$

E.g. $\left[\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right] = 5$.

a. Program the *trace* function in Eigenmath. (Do not use *contract* ;)

b. Let A, B be arbitrary matrices of type 2×2. Show with the help of Eigenmath the *Multiplication theorem for determinants*:

$$det(A * B) = det(A) \cdot det(B)$$

Test on self-chosen examples!

c. Let A, B be arbitrary matrices of type 2×2. Show with the help of Eigenmath the *Addition theorem for determinants*:

$$det(A + B) = det(A) + det(B) + trac(adj(A) * B)$$

Remark: Unexpectedly, the addition theorem for determinants turns out to be more difficult than the multiplication theorem.
○ When does the naively expected formula $det(A+B) = det(A) + det(B)$ apply? Test on self-chosen examples!

c. Let M be an arbitrary 2×2 matrix and E the 2×2 identity matrix. Use Eigenmath to show the so-called Cayley theorem:

$$M^2 - trace(M) \cdot M + det(M) \cdot E = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$$

Check the validity of Cayley's theorem with self-chosen examples.

The trace of a matrix and CAYLEY's theorem will soon play a crucial role in our study of the classification of linear and affine mappings of the plane.

$\bowtie$

In LINDNER [10, pp. 71–73] we show the use of determinants and the CRAMER rule using simplifying general methods of Geometric (CLIFFORD) Algebra.

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⋈

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