2 CheatSheet APOS Theory and A.C.E. Teaching Cycle

The social-constructivist APOS theory by Ed DUBINSKY et al. is used as a *theoretical research approach,* for practical *curriculum development* and as a computer-based didactic *teaching/learning method*. It sees itself as a continuation of PIAGET's developmental psychological level theory and is specifically dedicated to advanced mathematical thought processes.

1 The APOS theory

begins with the theoretical reflection of a mathematical object (topic) with the aim of a *genetic decomposition* of the concept in question. Specific mental constructions are suggested that a learner needs to acquire the concept. The realization then takes place through the design and implementation of a suitable sequence of instructions to stimulate the learner's reflection in a social (cooperative, CAS supported) context. *Mathematical understanding* begins with the manipulation of previously constructed mental or physical objects to form *actions*; these actions are interiorized ("internalized") into *process*es, which are finally encapsulated in the form of *object*s. Such objects can be decapsulated back to the processes from which they were formed. Processes or objects are organized in the form of *schema*s:

- **Action:** Instructions should be based on *constructive activities* (‚actions') of the learner.
  Example: the remainder class group (Z_{20}; $+_{20}$) and the subgroup H:={0,4,8,12,18}.
  The action concept is "start with 2 and add 4" to form the term "remainder class" 2+H.

- **Process:** The individual reflects and interiorizes actions; it can then carry out these actions in a controlled manner.
  Example: a *process understanding* of the concept of a remainder class consists in thinking of the associated process of forming sets, i.e. the operation of a fixed element x from Z_{20} on every element h of subgroup H. But: *the actual execution of this operation is no longer necessary* - instead, in a kind of thought experiment, it is only thought that it has been carried out.

- **Object:** An individual thinks of a process as an object when reflecting on operations, that act on that process as a whole.
  Example: the action concept of 2+H is reified into an object "2+H"; then one can speak of calculating with classes as separate objects ("formation of quotients") etc. The object 2+H can be decapsulated again to the set {2,6,10,14,18} if necessary.

- **Schema**: The systematic linking of objects and processes.
  Example: Groups such as (Z_{20}; $+_{20}$), rings etc. are organized in the schema of algebraic structures.

In the APOS theory, *learning difficulties* are usually explained with an *unsuccessful interiorization* of actions into processes or the *failed encapsulation* of processes into objects.

2 The A.C.E Teaching/Learning cycle

implements the didactic APOS theory approach as a methodical instruction strategy in the classroom.

- **Activities:** The class works in the social form of cooperative couples on the PC (e.g. in a laboratory) and carries out actions, that are intended to promote the formation of the desired mental constructions according to the genetic decomposition from the APOS analysis.
- **Class:** works in the form of cooperative teams (without a PC) in the classroom and works on paper & pencil problems that build on the experimental experiences from the A. phase on the PC.
- **Exercises:** are productive exercises following the A.C. phase; intended as traditional homework to be solved with/without PC/CAS support.

According to the APOS theory, knowing or being able to do something in mathematics means the following:
"An individual's mathematical knowledge is his/her tendency to approach mathematical problem situations through reflection in a social context."

Translated into English (wL) 2/2021

Example (function term):

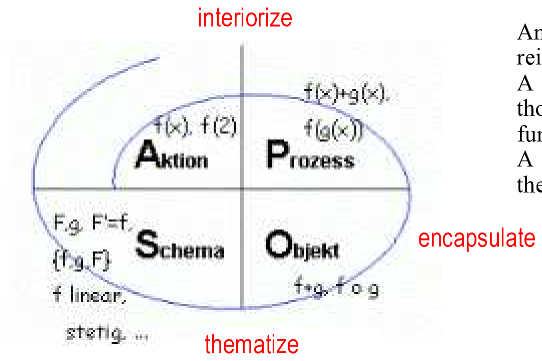

Fig. 1: The learning process according to the APOS theory using the example "function term *f*(*x*)".
An *action* or process concept "*calculate function value f (x)* e.g. *f*(2)" is reified into an object "*function*" *f*.
A function *graph* is then thought of as a whole thing and not originated thought single points. Only after that one can speak of operations with function objects e.g. chaining, derivation, integration etc.
A systematic linking of objects (f, F) and processes is e.g. addressed in the *schema* of the fundamental theorem of calculus.

Overview:

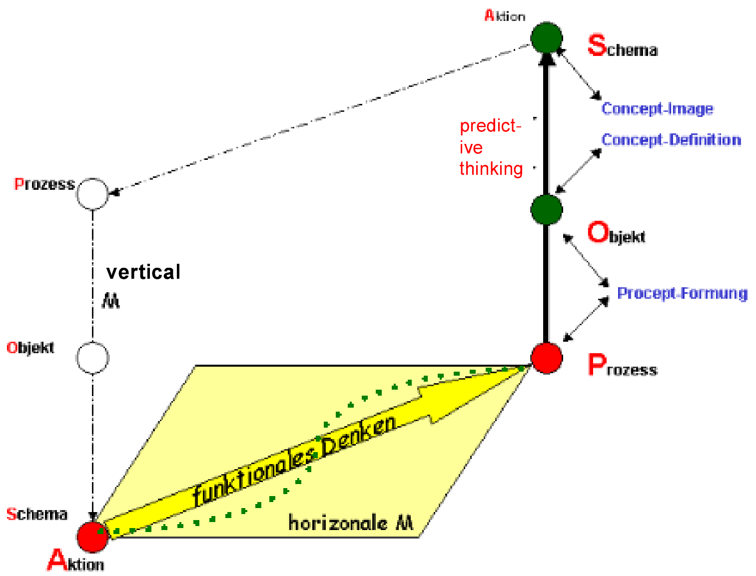

Fig. 2: According to the *RME*, the action-process-object-scheme learning process may be subdivided into a phase of *horizontal mathematics* (A-P) and a phase of *vertical mathematics* (O-S).

The *individual learning trajectories* (dotted, *green*) f the learners meander around the hypothetical learning trajectory (visualized as arrow from A to P, *yellow*) of the instructor in the A.P. phase.

An *O* understanding of a mathematical concept can be interpreted as a *concept definition* and an *S* understanding as a *concept image* in terms of TALL/VINNER.

RME (NL): Realistic Math Education project

functionales Denken (D) =
functional thinking (E)

References

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[2] DUBINSKY, E., and TALL, D.: Advanced Mathematical Thinking and the Computer.
In ***Advanced Mathematical Thinking***, D. Tall, Ed., vol. 5. Kluwer, Dordrecht, 1991, pp. 231–250.

[3] TALL, D., and VINNER, S.: Concept Images and Concept Definition in Mathematics with Particular reference to Limits and Continuity. ***Educational Studies in Mathematics,*** 12 (1991), 49–63